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B.Sc. Part II (Hons)
4th Paper

Linear Diff Equations with constant coefficients

Q Solve $(D^4 + 2D^2 + 1)y = x^2 \cos x$

Soln → For CF, $D^4 + 2D^2 + 1 = 0$

$\Rightarrow (D^2+1)^2 = 0 \Rightarrow D^2 + 1 = 0$

$\Rightarrow D = \pm i, \pm i$

$\therefore CF = (c_1 + c_2 x) e^{ix} + (c_3 + c_4 x) e^{-ix}$

$= (c_1 + c_2 x)(\cos x + i \sin x)$
$+ (c_3 + c_4 x)(\cos x - i \sin x)$

$= \cos x \left[(c_1 + c_3) + x(c_3 + c_4)\right]$
$+ \sin x \left[(c_1 - c_3) + ix(c_2 - c_4)\right]$

$\therefore CF = (A + Bx)\cos x + (C + Dx)\sin x$

where $A = c_1 + c_3$, $B = c_3 + c_4$
$C = c_1 - c_3$, $D = i(c_2 - c_4)$

For PI

$$PI = \frac{1}{(D^4 + 2D^2 + 1)} x^2 \cos x$$

$\therefore \quad PI = \frac{1}{(D^2+1)^2} x^2 \cos x$

First we integrate $\cos x$.

Putting $D^2=-a^2$ makes denominator Zero.

So, we take $\cos x$ = Real part of e^{ix}.

$\therefore \quad PI = R.P. \text{ of } \frac{1}{(D^2+1)^2} x^2 e^{ix}$

$\Rightarrow PI = R.P. \text{ of } e^{ix} \frac{1}{[(D+i)^2+1]^2} x^2$

$= R.P. \text{ of } e^{ix} \frac{1}{(D^2+2Di)^2} x^2$

$= RP \text{ of } e^{ix} \frac{1}{\left[2Di\left(1+\frac{D}{2i}\right)\right]^2} x^2$

$= RP \text{ of } -\frac{e^{ix}}{4} \frac{1}{D^2} \left(1+\frac{D}{2i}\right)^{-2} x^2$

$= RP \text{ of } -\frac{e^{ix}}{4} \frac{1}{D^2} \left[1-\frac{2D}{2i}+\frac{3D^2}{4i^2} - \dots\right] x^2$

$= R.P \text{ of } -\frac{e^{ix}}{4} \frac{1}{D^2} \left[x^2 - \frac{2 \times 2x}{2i} - \frac{3 \times 2}{4} - 0\right]$

$= R.P \text{ of } -\frac{e^{ix}}{4} \frac{1}{D^2} \left[x^2 + 2ix - \frac{3}{2}\right]$

$\therefore PI = R.P \text{ of } -\frac{e^{ix}}{4} \frac{1}{D} \int \left(x^2 + 2ix - \frac{3}{2}\right) dx$

$\Rightarrow PI = R.P \text{ of } -\frac{e^{ix}}{4} \frac{1}{D} \left[\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right]$

$\Rightarrow PI = R.P. \text{ of } -\frac{e^{ix}}{4} \int \left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right) dx$

$\Rightarrow PI = R.P. \text{ of } -\frac{e^{ix}}{4} \left[\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right]$

$\Rightarrow PI = RP \text{ of } -\frac{1}{4} (\cos x + i \sin x) \cdot \left(\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right)$

$\Rightarrow PI = \cancel{R.P. \text{ of}} -\frac{1}{4} \left[\left(\frac{x^4}{12} - \frac{3}{4}x^2\right) \cos x - \frac{x^3}{3} \sin x\right]$

$\therefore$ complete solution is given by

$y = CF + PI$

$\Rightarrow y = (A + Bx) \cos x + (C + Dx) \sin x - \frac{1}{4} \left[\left(\frac{x^4}{12} - \frac{3}{4}x^2\right) \cos x - \frac{x^3}{3} \sin x\right]$